Multifractal characterization of the spatial distribution of ulexite in a Bolivian salt flat

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Understanding spatial patterns is a critical and under-explored aspect of remote sensing. This paper describes how multifractal theory can be applied to characterize these heterogeneous patterns in remotely sensed data as well as to determine the operational scale. An example based on the characterization of ulexite distribution on the world’s largest salt flat (10 000 km²), located in Bolivia, using a binarized Landsat Thematic Mapper (TM) 4/7 ratio image, is used to describe the step-by-step procedure. Distribution was well characterized by the multifractal parameters and expressed through the \( f-a \), \( \tau-q \) and \( D-q \) relationships. Moments from \( q = -2 \) to 5 showed a linear trend in scales from approximately 0.007 to 10 000 km². This implies that the attribute analysed could be measured at different scales, within defined boundaries, and up- and down-scaled using the multifractal parameters. In addition, the asymmetry shown by the \( f-a \) spectrum indicates the presence of clusters with high probability of finding ulexite, and large areas where the mineral might be found in small patches. The areas with a high probability of finding ulexite were mapped to guide any future field survey. Using the maximum entropy concept, the operational scale to determine the mineral was obtained at 1062 m.

1. Introduction

Geophysical and geographical systems are characterized by extreme spatial and temporal variabilities. These constitute major impediments, both conceptually and methodologically, to advancing all sciences that use geographic information (Antle et al. 1999). This is of particular importance when studies deal with different geospatial attributes, which operate at different scales.

Application of mono-fractal theory to geospatial data has been traditionally associated with fractal sets and a unique scaling exponent (specifically the fractal dimension, \( D \)). This dimension describes a uniform set (uniform distribution of a measure) or a homogeneous system, but does not provide information about either a distribution of low and high irregularities of the measure within a set nor about its spatial variability. Most fractal applications in remote sensing also use a mono-fractal approach, which assumes that the spatial distribution of the measured attribute can be characterized by a single fractal dimension. For example, Keller et al. (1989) and Pentland (1984) used the single fractal dimension technique to assess
landscape complexity and Lovejoy (1982) calculated the perimeter–area relation of land areas and clouds. Rees (1992) and Ricotta and Avena (1998) used the fractal dimension technique to perform texture analysis on satellite images. Very special types of scale-changing operations were considered and only self-similar and, occasionally, self-affine transformations have been employed. In the former case, the small scales are reduced carbon copies of the large one; in the latter case, ‘squashing’ or ‘compression’ along coordinate axes is also allowed (Lavallée et al. 1993).

Many systems presenting non-linear variation among scales might be a consequence of fields generated in cascade processes. Those processes may produce fields, exhibiting multifractal or multiscaling behaviours, which are characterized by spatial or temporal non-linear scaling exponents with respect to the statistical moment. In such a case, the parameters generated could be used to make inferences about the properties of the system at different scales.

The application of multifractal theory to remotely sensed data is incipient, e.g. Dubayah et al. (1997), Pecknold et al. (1997), Cheng (1999), Parrienello and Vaughan (2002). Multifractal scaling might be widely applied as a result of developments such as the generalized scale invariance (Schertzer and Lovejoy 1983, 1985). In particular, generalized scale-invariant systems can include fields rather than sets (i.e. multifractal measures); also, the types of possible scale transformations encompass not only differential stratification (self-affinity) but also rotation and anisotropy, which vary from one place to another (Lavallée et al. 1993).

An additional feature of the application of multifractal theory to geospatial data is the capacity to calculate the entropy dimension, which is related to Shannon entropy (Shannon and Weaver 1949). The entropy dimension allows for the assessment of the operational scale of each attribute under study.

In this paper, we provide a step-by-step description of an estimation of multifractal parameters applied to remotely sensed data. The goal is to show how useful the multifractal analysis could be to investigate spatial features in an image and to estimate the operational scales of each attribute under study. As an example, the analysis was applied to a Landsat Thematic Mapper (TM) image of the salt flat (salar) of Uyuni, located in Western Bolivia. This application is based on the hypothesis that ulexite, the principal borate mineral in the salar, should have high values of TM 4/7 ratio compared with halite or rock salt, which constitutes more than 90% of the crust (Sabins 1997).

2. Theory

2.1 Multifractals

It is now widely accepted that physical systems that exhibit chaotic behaviour are generic in nature. Since these systems lose information exponentially fast it is possible to follow and predict their motion in any detail only for short timescales (Chhabra et al. 1989). To describe their long-term dynamical behaviour, one must resort to suitable statistical descriptions. One such description is multifractal formalism (Hentschel and Procaccia 1983, Halsey et al. 1986, Chhabra et al. 1989, Chhabra and Jensen 1989). Multifractal theory permits the characterization of complex phenomena in a fully quantitative fashion, for both temporal and spatial variations. Multifractal techniques and notions are increasingly widely recognized as the most appropriate and
straightforward framework within which to analyse and simulate not only the scale dependency of the geophysical observables, but also their extreme variability over a wide range of scales (Schertzer and Lovejoy 1994).

The basic equation of fractal theory expresses the relationship between the number and the size of the objects (Feder 1988):

$$ N(\varepsilon) \sim \varepsilon^{-D_0} $$

(1)

where $N(\varepsilon)$ is the number of objects, $\varepsilon$ is the scale and $D_0$ is the fractal dimension. The box-counting technique is used to estimate the scaling properties of a set by covering the set with boxes of size $\varepsilon$ and counting the number of boxes containing at least one pixel representing the object under study:

$$ D_0 = - \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(\varepsilon)} $$

(2)

Provided the limit exists, the infinitum of $N(\varepsilon)$ is approximated by varying the origin of the grid until the smallest number is found. Using equation (2), the box-counting dimension $D_0$ can be determined as the negative slope of $\log N(\varepsilon)$ versus $\log(\varepsilon)$, measured over a range of box widths.

On the other hand, the fractals found in nature (also called physical fractals) are statistically self-similar only in a defined range of scales (Feder 1988). That is, a real fractal object in nature has a finite size, and this inevitably introduces a lower and an upper cut-off point onto the scale on which fractal scaling can be observed. Numerical techniques, commonly used to obtain the fractal dimension of such physical fractals, are described in Vicsek (1992).

In a homogeneous system, the probability ($P$) of a measured quantity (measure) varies with scale $\varepsilon$ as (Chhabra et al. 1989, Evertsz and Mandelbrot 1992, Vicsek 1992):

$$ P(\varepsilon) \sim \varepsilon^D $$

(3)

where $D$ is the fractal dimension. For heterogeneous or non-uniform systems the probability within the $i$th region $P_i$ varies as:

$$ P_i(\varepsilon) \sim \varepsilon^z_i $$

(4)

where $z_i$ is the Lipschitz–Hölder exponent or singularity strength, characterizing scaling in the $i$th region or spatial location (Feder 1988). The parameter $z_i$ quantifies the degree of regularity in point $x_i$. Loosely speaking, any measure $\mu$ of an interval $[x_i, x_i+\Delta x]$, behaves as $(\Delta x)^{z_i}$ (Halsey et al. 1986). For a uniform distribution one finds $z_i(x)=1$ for all $x$. More generally, for any real value $a>0$ the distribution with density $x^{-a-1}$ on $[0, 1]$ has $z_i(0)=a$ and $z_i(x)=1$ for all $x \in (0, 1]$. Values $z_i(x)<1$ indicate, thus, a burst of the event around $x$ ‘on all levels’, while $z_i(x)>1$ is found in regions where events occur sparsely (Riedi 1999). Similar $z_i$ values might be found at different positions in the space. The number of boxes $N(x)$ where the probability $P_i$ has singularity strengths between $x$ and $x+dx$ is found to scale as (Chhabra et al. 1989, Halsey et al. 1986):

$$ N(x) \sim \varepsilon^{-f(x)} $$

(5)
where \( f(x) \) can be considered as the generalized fractal dimension of the set of boxes with singularities \( x \) (Kohmoto 1988). The exponent \( x \) can take on values from the interval \([-\infty, +\infty]\), and \( f(x) \) is usually a function with a single maximum at \( df[x(q)]/dx(q) = 0 \) (where \( q \) is the order moment of a statistic distribution). When \( q = 0 \), \( f_{\text{max}} \) is equal to the box-counting dimension, \( D_0 \) (Vicsek 1992, Gouyet 1996).

Multifractal sets can also be characterized on the basis of the generalized dimensions of the \( q \)th order moment of a distribution, \( D_q \), defined as (Hentschel and Procaccia 1983):

\[
D_q = \lim_{\varepsilon \to 0} \left( \frac{1}{q-1} \frac{\log \mu(q, \varepsilon)}{\log(\varepsilon)} \right)
\]

where \( \mu(q, \varepsilon) \) is the partition function (Chhabra et al. 1989):

\[
\mu(q, \varepsilon) = \sum_{i=1}^{N(\varepsilon)} P_i^q(\varepsilon)
\]

The generalized dimension \( D_q \) is a monotonic decreasing function for all real \( q \)s within the interval \([-\infty, +\infty]\). When \( q < 0 \), \( \mu \) emphasizes regions in the distribution with less concentration of a measure, whereas the opposite is true for \( q > 0 \) (Chhabra and Jensen 1989).

Also, the partition function scales as:

\[
\mu(q, \varepsilon) \sim \varepsilon^{\tau(q)}
\]

where \( \tau(q) \) is the correlation exponent of the \( q \)-th order moment defined as (Halsey et al. 1986, Vicsek 1992):

\[
\tau(q) = (q - 1) D_q
\]

The connection between the power exponents \( f(x) \) (equation (5)) and \( \tau(q) \) (equation (9)) is made via the Legendre transformation (Callen 1985, Halsey et al. 1986, Chhabra and Jensen 1989):

\[
f(x(q)) = q x(q) - \tau(q)
\]

and

\[
x(q) = \frac{d \tau(q)}{dq}
\]

\( f(x) \) is a concave downward function with a maximum at \( q = 0 \). When \( q \) takes the values of \( q = 0, 1 \) or \( 2 \), equation (6) is reduced to:

\[
D_0 = \lim_{\varepsilon \to 0} \frac{\log(N(\varepsilon))}{\log(\varepsilon)}, \quad D_1 = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N(\varepsilon)} \mu_i(\varepsilon) \log(\mu_i(\varepsilon))}{\log(\varepsilon)}, \quad D_2 = \lim_{\varepsilon \to 0} \frac{\log(C(\varepsilon))}{\log(\varepsilon)}
\]

respectively, with \( C(\varepsilon) \) being the correlation function (Grassberger and Procaccia 1983).

The values \( D_0 \), \( D_1 \) and \( D_2 \) are known as the capacity dimension, the entropy dimension and the correlation dimension, respectively. The capacity dimension provides global (or average) information about a system (Voss 1988). The entropy dimension is related to the information (or Shannon) entropy (Shannon and Weaver...
The correlation dimension $D_2$ is mathematically associated with the correlation function (Grassberger and Procaccia 1983) and computes the correlation of measures contained in a box of size $\varepsilon$ (Theiler 1987). The relationship between $D_0$, $D_1$, and $D_2$ is

$$D_2 \leq D_1 \leq D_0$$  \hspace{1cm} (12)$$

where the equality $D_0=D_1=D_2$ occurs only if the fractal is statistically or exactly self-similar and homogeneous (Korvin 1992).

### 2.2 Information entropy

The information dimension ($D_1$) is a measure of information entropy, as defined in equation (11). $D_1$ is the slope of the line described by

$$\sum_{i=1}^{N(L)} \mu_i(L) \log[\mu_i(L)] \text{ vs } \log(L)$$  \hspace{1cm} (13)$$

The information entropy has been extensively studied in communication theory (Andraud et al. 1997).

Shannon and Weaver (1949) defined the information of the system by its entropy. If a system has $N$ different possible states with probability of occurrence $p_i$, $i=1,2\ldots,N$, then the gain in information from observing the occurrence of the event $(i)$ is defined as:

$$I_i = \ln \frac{1}{p_i}$$  \hspace{1cm} (14)$$

This definition follows from the fact that for two independent events with probabilities $p',q'$ one has $I_{p',q'}=I_{p'}+I_{q'}$; from the fact that for certain event with $p'=1$ we have $I_1=0$; and from the requirement $I\geq0$.

The expected value of such a gain in information is defined as the entropy ($H$) of the system:

$$H = \sum_{i=1}^{N} p_i I_i = - \sum_{i=1}^{N} p_i \ln p_i$$  \hspace{1cm} (15)$$

where $p_i$ is the probability that the system assumes its $i$th possible outcomes (Williams 1997).

Information entropy is also a measure of the unpredictability of a system. For a uniform probability, $p_i=1/N$ and $H$ is at its maximum.

$$p_i = \frac{1}{N} \Rightarrow H = - \sum_{i=1}^{N} \left( \frac{1}{N} \right) \ln \left( \frac{1}{N} \right) = - N \left( \frac{1}{N} \right) \ln \left( \frac{1}{N} \right) = \ln(N)$$  \hspace{1cm} (16)$$

If $H$ is maximum we have a totally disordered system (maximum uncertainty). By contrast, if $H=0$ the system has maximum predictability (it is totally ordered). $H$ varies between 0 and $\ln(N)$.

When $H$ is associated with the observation scale ($L$), it determines a characteristic size having maximum entropy, i.e. the optimum scale at which the system must be observed and characterized (Beghdadi et al. 1993, Andraud et al. 1997).
3. Methods

3.1 Image processing

A Landsat TM image of Salar de Uyuni, south-west Bolivia (figure 1(a)) was used to demonstrate the estimation of multifractal parameters on a remotely sensed image. Salar de Uyuni is the world’s largest salar (10 000 km$^2$), which is known to contain borate minerals. Risacher (1989) analysed brine samples from 68 shallow drill holes and prepared a map of boron concentration and spatial distribution at a resolution of about 147 km$^2$, which constitutes a very sparse sampling. The Landsat TM spans the salar with more than $11 \times 10^6$ ground cells, each of which represents 0.0009 km$^2$. The Bolivian government contracted an international company (Intercontinental Resources, Inc.) to conduct a Landsat evaluation of the salar. The study showed that the reflectance spectrum of ulexite (NaCaB$_5$O$_8$H$_2$O: figures 11–16 in Sabins (1997)), which is the principal borate mineral in the salar, differed from the spectrum of halite (NaCl), which constitutes 90% of the crust and that the TM 4/7 ratio could be useful for ulexite exploration (Sabins 1997). These two studies were used as reference to interpret the results of the example on multifractal analysis. No field verification was conducted, since our objective has been to show how multifractal parameters can characterize the spatial distribution of a measure and not the ulexite quantification per se.
Binarization was done using threshold filtering in the range 45–98 of the grey level, which corresponded to the range of values encountered when the pixels near the sampling sites conducted by Risacher (1989) were referenced into the TM 4/7 ratio image. The multifractal analysis was conducted on the binarized image.

The Landsat TM image was geometrically corrected using mapping polynomials and 15 ground control points from the 1:100 000 photogrammetry map of Bolivia. The grey levels of the TM 4/7 ratio image (figure 1(b)) were converted into a binary image using the MatLab image tool (figure 1(c)).

3.2 Determination of multifractal parameters

The method used was that developed by Chhabra and Jensen (1989) to calculate the \( f(q) \) spectrum, and implemented using the MatLab package, as described by Posadas et al. (2003). The spatial distribution of ulexite concentration was partitioned in boxes of size \( L \), for \( L = 1, 2, 4, 8, 20, 25, 50, 100, 200 \) and 400 pixels. The normalized measure \( \mu_i(q, L) \) was calculated for values of \( q \) that varied in steps of 0.5:

\[
\mu_i(q, L) = \frac{P_i^q(L)}{\sum_{i=1}^{N(L)} P_i^q(L)}
\]

where \( P_i(L) \) is the fraction (or probability) of ulexite contained in each \( i \)th box of size \( L \). The multifractal spectrum, \( f(z(q)) \) vs \( z(q) \), was calculated as (Chhabra and Jensen 1989, Chhabra et al. 1989):

\[
f(q) = -\lim_{N \to \infty} \frac{1}{\log(N)} \sum_{i=1}^{N(L)} \mu_i(q, L) \log[\mu_i(q, L)]
\]

\[
z(q) = -\lim_{N \to \infty} \frac{1}{\log(N)} \sum_{i=1}^{N(L)} \mu_i(q, L) \log[P_i(L)]
\]

Since the choice of an appropriate scale range is a crucial step in multifractal analysis (Saucier and Muller 1999), the maximum value of \( L \) and \( q \) that can be used in equations (18) and (19) was assessed by the linear behaviour of the function

\[
\sum_{i=1}^{N(L)} \mu_i(q, L) \log[\mu_i(q, L)] \text{ vs } \log(L) \quad \text{and} \quad \sum_{i=1}^{N(L)} \mu_i(q, L) \log[P_i(L)] \text{ vs } \log(L)
\]

for all the \( q \)s considered. The moments \( (q) \) ranged from \(-2\) to 5 with step variation of 0.5, which is fine enough to show the multifractal behaviour in the very narrow range of \( q \)s. The linearity was accepted only if the coefficient of determination \( (R^2) \) of a linear fit was above 0.95. In addition, we tested the validity of the results by verifying that the tangent of the graph \( f(z) \) vs \( z \) at \( z(q=1) \) is the bisector defined by \( df(z)/dz=q \). The point of intersection corresponds to \( f[z(1)]=z(1)=D_1 \) (Evertsz and Mandelbrot 1992). The generalized fractal dimensions \( D_q \) and the mass exponent \( \tau(q) \) were obtained with equation (9) and equation (10), respectively.

The behaviour of the mass exponent \( \tau(q) \) vs \( q \) was also analysed. If \( \tau(q) \) behaves non-linearly with respect to \( q \), the measure (\( \mu \)) is said to be multifractal.
The size of the entire image considered for the multifractal analysis was 1200 × 1200 pixels (or 1296 km²). The first step in the analysis was the assessment of the behaviour of the generalized entropy as a function of the scale for different moments $q$ (Posadas et al. 2003). The $q$ values for which the system showed a linear behaviour constituted the boundaries within which the multifractal analysis was conducted.

3.3 Shannon entropy and cluster classes

The entropy $(H)$, as described in equation (15), was calculated with mathematical morphology toolbox software (SDC 2002) developed for Matlab. Another Matlab program was also used to obtain the cluster classes as well as the cumulative percentage of ulexite area concentration.

4. Results and discussion

4.1 Multifractal parameters

The results of the multifractal analysis are shown in figure 2(a)–(d). Figure 2(a) shows the linear behaviour of the system. These results identify the range of moments that need to be considered in order to study the scale variation of ulexite concentrations throughout the salar. Moments from $q = -2$ to 5 showed a linear trend. The coefficients of determination ($R^2$) for fitting $\Sigma_{j} \mu_j \times \log(\mu_j) \times \log(L)$ ranged from 0.98 to 0.99 for all the studied $q$. This linear behaviour (from scales of

Figure 2. (a) Generalized entropy vs log($L$) for the Salar de Uyuni image showing the linear behaviour for the range of moments from $q = -2$ to $q = 5$. (b) Multifractal spectrum, $f(\alpha)$ vs $\alpha$ for the binary image of Salar de Uyuni. (c) Generalized fractal dimension for the Salar de Uyuni image. (d) Mass exponent showing a discontinuous line.
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4.2 Information entropy and cluster classes

The variation of the entropy \((H)\) as a function of the scale \((L)\) expressed in \(\text{m}^2\) is shown in figure 3. Entropy is maximized at a scale of \(1062\ \text{m}^2\), the operational scale

approximately \(0.007\ \text{km}^2\) up to \(10000\ \text{km}^2\) indicates that the properties analysed in the salar system (ulexite) could be measured at different scales within defined boundaries and up-and down scaled using ‘multi’ fractal parameters (Pecknold et al. 1997).

The \(f(z)\) spectrum (figure 2(b)) shows an asymmetrical curve with one peak and a greater tendency towards the right side, where \(q<0\). The magnitude of the change around the maximum values of \(f[\alpha(0)]\) is a measure of the symmetry of the \(f(z)\) spectrum. The differences \((D_0-D_1)\) and \(\{D_0-f[\alpha(-1)]\}\) indicate the deviation of the \(f(z)\) spectrum from its maximum value \((q=0)\) toward the left side \((q>0)\), and the right side \((q<0)\) of the curve, respectively. Asymmetry toward the left site indicates domination of large or pressure of extremely large values in the spatial variability pattern while asymmetry to the right indicates domination of small or presence of extremely small values (Eghball et al. 2003). Our results for these parameters were \(f[\alpha(0)]=D_0=1.656; (D_0-D_1)=0.11\) and \(\{D_0-f[\alpha(-1)]\}=0.51\), indicating a greater degree of asymmetry towards the right \((q<0)\) than to the left \((q>0)\). This information is very important since the values of \(q>0\) are directly associated with the measure that, in our example, is associated with the spatial concentration of ulexite. Thus, the asymmetry indicates domination of small values in the spatial variability. In fact \(87\%\) of the area is dominated by small values. Small cluster of pixels with high ulexite levels, embedded within areas with no ulexite, caused this asymmetry. If one is interested in locating the spatial position from which to extract the ulexite, it is possible to explore using the highest values of \(q\), thus selecting the places which contain the greatest concentration of ulexite, since large values of \(q\) magnify the presence of the measurement (Appleby 1996).

The \(f(z)\) function presents finite values, i.e. it does not intersect the \(z\) axis (figure 2(b)). From the maximum point to the left, the function describes the part of the image with high ulexite concentration. The fact that the \(f(z)\) value for the largest positive \(q\) tends to be near the maximum indicates that the ulexite concentration is clustered in compact areas, rather than being ubiquitously distributed. This occurs when the slope of the curve \(D(q)\) vs \(q\) is near 0 as shown in figure 2(c). This also corresponds to \(\tau(q)\) becoming linear, as shown in figure 2(d) (for \(q>0\)).

The generalized fractal dimension \(D_q\) shown in figure 2(c) corroborates the differential pattern for \(q>0\) (almost a constant line) and \(q<0\) (a line with an accentuated inclination) made evident in figure 2(b). Here, an almost constant line for \(q>0\) is typical of a distribution for homogeneous systems, whereas an inclined line for \(q<0\) represents a system with heterogeneous distribution. The mass exponent shown in figure 2(d) represents a curved line with a concavity change in the vicinity of \(q=0\), which also shows heterogeneity within the system for values of \(q<1\) presenting a steeper slope as \(q\) tends to more negative values.

The multifractal parameters give a good description of what can be seen in the image: concentrated areas with high ulexite concentrations near the edges of the salar and small spots elsewhere.
at which these types of studies should be conducted. According to the theory (Andraud et al. 1997, Van Siclen 1997), this should also guide the sampling scheme for future ground measurement studies.

The map shown in figure 4(a), (b) might be used to guide any effort of ground measurements. There is a compact continuous cluster in the northern border of the salar that contains 74.8% of all the pixels that exhibited a high TM 4/7 ratio. There are three other clusters that contain, respectively, 9.6%, 3.8% and 3.7% of all the pixels. Combined, these four clusters represent around 13% of the area of the salar, around 1300 km².

Figure 3. Entropy information ($H$) from Salar de Uyuni as a function of scale.

Figure 4. (a) Map showing the clusters’ classification according to their size representing 74.8%, 9.6%, 3.8% and 3.7% for shaded squares, shaded vertical lines, shaded diagonal lines, and shaded horizontal lines, respectively. (b) Cluster classes as a function of the cumulative area (%) showing the three largest clusters containing borate as shown in (a).
5. Conclusions

The study reported in this paper showed that the multifractal parameters clearly summarized the behaviour of the binary distribution of high TM 4/7 ratio, which may indicate concentrations of ulexite. From the maximum of the \( f(a) \) spectrum to the left \( (q>0) \) there is little asymmetry, indicating the presence of large clusters of pixels containing a high TM 4/7 ratio. An inspection of the image shows that near the salar borders there is indeed a compact band of pixels with a high probability of containing ulexite minerals. In turn, the right portion of the spectrum \( (q<0) \) showed greater asymmetry. This asymmetry reflects the dominance of small values of the TM 4/7 ratio. Small islands with a high TM 4/7 ratio occurring within large areas with low TM 4/7 ratio can be seen in the image, thus confirming the interpretation of the features described by the \( f(a) \) spectrum. The presence of large TM 4/7 ratio clusters was confirmed by the sudden ending of the \( f(a) \) trajectory at a large \( q \) \( (q=5) \) and also evidenced by a flat response for \( D_q \) when \( q>0 \) and the linear response of \( q \) for \( q>0 \).

The entropy \( H \) showed a quadratic response with a maximum at a pixel size of around 1.1 km. This maximum defines the operational scale at which the phenomenon operates (Bian 1997).

The results presented demonstrated the usefulness of multifractal techniques in extracting features from remotely sensed data as well as their potential in exploratory studies exemplified with the determination of the distribution of ulexite in the salar de Uyuni.

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Appendix. Lists of symbols

- \( \alpha_i \) : Singularity strength that characterizes the scale in the \( i \)th region;
- \( f(a) \) : generalized fractal dimension;
- \( D_0 \) : capacity dimension;
- \( P_i \) : probability distribution;
- \( \varepsilon \) : scale;
- \( N(\varepsilon) \) : number of objects of size \( \varepsilon \);
- \( D_q \) : generalized fractal dimension;
- \( q \) : statistics moment;
- \( \mu \) : mass or measure associated with probability \( P_i \);
- \( \tau \) : mass or correlation exponent;
- \( D_1 \) : entropy dimension;
- \( D_2 \) : correlation dimension;
- \( C(\varepsilon) \) : correlation function;
- \( L \) : scale used to the multifractal analysis;
- \( H \) : information function;
- \( R^2 \) : coefficient of determination;
- \( x \) : position of a point;
- \( \Delta x \) : interval of variation of \( x \);
- \( a \) : any real parameter larger than zero.
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